

GEOMETRY VIA COHERENT STATES

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Abstract

It is shown how the coherent states permit to find different geometrical objects as the geodesics, the conjugate locus, the cut locus, the Calabi's diastasis and its domain of definition, the Euler-Poincaré characteristic, the number of Borel-Morse cells, the Kodaira embedding theorem.

1 Coherent state manifold and coherent vector manifold

In this talk the coherent states \mathbb{C} are presented as a powerful tool in global differential geometry and algebraic geometry \mathbb{C} , \mathbb{C} . For general references and proofs see also \mathbb{C} , \mathbb{C} . The results are illustrated on the complex Grassmann manifold \mathbb{C} .

We start with some notation. Let π be a unitary irreducible representation, G a Lie group, \mathcal{H} a separable complex Hilbert space. Let also the orbit $\widetilde{\mathbf{M}} = \tilde{\pi}(G)\tilde{\psi}_0$, where $\psi_0 \in \mathcal{H}$, $\xi : \mathcal{H} \rightarrow \mathbf{P}(\mathcal{H})$ is the projection $\xi(\psi_0) = \tilde{\psi}_0$. Then there is a diffeomorphism $\widetilde{\mathbf{M}} \approx G/K$, where K is the stationary group of $\tilde{\psi}_0$. If $\iota : \widetilde{\mathbf{M}} \hookrightarrow \mathbf{P}(\mathcal{L})$ is a biholomorphic embedding in some projective Hilbert space, then $\widetilde{\mathbf{M}}$ is called *coherent state manifold*. If $\sigma : \widetilde{\mathbf{M}} \rightarrow \mathcal{S}(\mathcal{H})$ is a local section in the unit sphere in \mathcal{H} , let $\mathbf{M}' = \sigma(\widetilde{\mathbf{M}})$ be the holomorphic line bundle associated to the principal holomorphic bundle $P \rightarrow G^c \rightarrow G^c/P$ by a holomorphic character χ of the parabolic subgroup P of the complexification G^c of G . \mathbf{M}' is a quantization bundle over $\widetilde{\mathbf{M}}$ \mathbb{C} .

The following assertions are equivalent: there exists the embedding ι ; there exists a positive line bundle \mathbf{M}' over $\widetilde{\mathbf{M}}$; the line bundle \mathbf{M}' is ample; there exists m_0 such that for $m \geq m_0$, $\mathbf{M} = \mathbf{M}'^m = \iota^*[1]$, where $[1]$ is the hyperplane bundle over $\mathbf{P}(\mathcal{L})$.

\mathbf{M} is called *coherent vector manifold*. The Perelomov's coherent vectors are

$$\mathbf{e}_{Z,j} = \exp \sum_{\varphi \in \Delta_n^+} (Z_\varphi F_\varphi^+)j, \quad \underline{\mathbf{e}}_Z = (\mathbf{e}_Z, \mathbf{e}_Z)^{-1/2} \mathbf{e}_Z, \quad (1)$$

$$\mathbf{e}_{B,j} = \exp \sum_{\varphi \in \Delta_n^+} (B_\varphi F_\varphi^+ - \bar{B}_\varphi F_\varphi^-)j, \quad \mathbf{e}_{B,j} := \underline{\mathbf{e}}_{Z,j}, \quad (2)$$

where j is an extreme weight vector, Δ_n^+ denotes the positive non-compact roots, $Z := (Z_\varphi) \in \mathbb{C}^d$ are local coordinates in the neighbourhood $\mathcal{V}_0 \subset \widetilde{\mathbf{M}}$ and $d = \dim_{\mathbb{C}} \widetilde{\mathbf{M}}$. In eqs. (1), (2) $F_\varphi^+ j \neq 0$, $F_\varphi^- j = 0$, $\varphi \in \Delta_n^+$, and $(\underline{\mathbf{e}}_{Z'}, \underline{\mathbf{e}}_Z)$ is the hermitian scalar product of holomorphic sections in the holomorphic line bundle \mathbf{M} over $\widetilde{\mathbf{M}}$.

2 Geodesics

Proposition 1 *For hermitian symmetric spaces, the dependence $Z(t) = Z(tB)$ appearing when one passes from eq. (2) to eq. (1) describes in \mathcal{V}_0 a geodesic.*

For example, on the Grassmannian $G_n(\mathbb{C}^{m+n}) = SU(n+m)/S(U(n) \times U(m))$,

$$\begin{aligned} G_n(\mathbb{C}^{m+n}) &= \exp \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} o = \begin{pmatrix} \cos \sqrt{BB^*} & B \frac{\sin \sqrt{B^*B}}{\sqrt{B^*B}} \\ -\frac{\sin \sqrt{B^*B}}{\sqrt{B^*B}} B^* & \cos \sqrt{B^*B} \end{pmatrix} o \\ &= \begin{pmatrix} \mathbb{1} & Z \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} (\mathbb{1} + ZZ^*)^{1/2} & 0 \\ 0 & (\mathbb{1} + Z^*Z)^{1/2} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -Z^* & \mathbb{1} \end{pmatrix} o, \end{aligned}$$

the geodesics in \mathcal{V}_0 are given by the expression

$$Z = B \frac{\text{tg} \sqrt{B^*B}}{\sqrt{B^*B}}. \quad (3)$$

3 Conjugate locus

Proposition 2 *For Hermitian symmetric spaces $\widetilde{\mathbf{M}}$, the parameters B_φ in formula (2) of normalised coherent states are normal coordinates in the normal neighbourhood \mathcal{V}_0 .*

Theorem 1 *Let $\widetilde{\mathbf{M}}$ be a Hermitian symmetric space parametrized in \mathcal{V}_0 as in eqs. (1), (2). Then the conjugate locus of the point o is obtained vanishing the Jacobian of the exponential map $Z = Z(B)$ and the corresponding transformations of the chart from \mathcal{V}_0 .*

Theorem 2 *The tangent conjugate locus C_0 of $\mathbf{O} \in G_n(\mathbb{C}^{m+n})$ is*

$$C_0 = \bigcup_{k,p,q,i} \text{ad } k(t_i H), \quad i = 1, 2, 3; \quad 1 \leq p < q \leq r, \quad k \in K; \quad (4)$$

$$H = \sum_{i=1}^r h_i D_{i n+i}, \quad h_i \in \mathbb{R}, \quad \sum h_i^2 = 1, \quad (5)$$

$$\begin{aligned} t_1 &= \frac{\lambda \pi}{|h_p \pm h_q|}, \quad \text{multiplicity } 2; \\ t_2 &= \frac{\lambda \pi}{2|h_p|}, \quad \text{multiplicity } 1; \\ t_3 &= \frac{\lambda \pi}{|h_p|}, \quad \text{multiplicity } 2|m-n|; \quad \lambda \in \mathbb{Z}^*. \end{aligned} \quad (6)$$

The conjugate locus of \mathbf{O} in $G_n(\mathbb{C}^{m+n})$ is given by the union

$$\mathbf{C}_0 = \mathbf{C}_0^W \cup \mathbf{C}_0^I, \quad (7)$$

$$\mathbf{C}_0^I = \exp \bigcup_{k,p,q} \text{Ad } k(t_1 H), \quad (8)$$

$$\mathbf{C}_0^W = \exp \bigcup_{k,p} \text{Ad } k(t_2 H). \quad (9)$$

Exponentiating the vectors of the type $t_1 H$ we get the points of \mathbf{C}_0^I (at least two of the stationary angles with \mathbf{O} are equal); $t_2 H$ are sent to the points of \mathbf{C}_0^W (at least one of the stationary angles with \mathbf{O} is 0 or $\pi/2$).

\mathbf{C}_0^W is given by the disjoint union

$$\mathbf{C}_0^W = \begin{cases} V_1^m \cup V_1^n, & n \leq m, \\ V_1^m \cup V_{n-m+1}^n, & n > m, \end{cases} \quad (10)$$

where

$$V_1^m = \begin{cases} \mathbb{CP}^{m-1}, & \text{for } n = 1, \\ W_1^m \cup W_2^m \cup \dots \cup W_{r-1}^m \cup W_r^m, & \text{for } 1 < n, \end{cases} \quad (11)$$

$$W_r^m = \begin{cases} G_r(\mathbb{C}^{\max(m,n)}), & n \neq m, \\ \mathbf{O}^\perp, & n = m, \end{cases} \quad (12)$$

$$V_1^n = \begin{cases} W_1^n \cup \dots \cup W_{r-1}^n \cup \mathbf{O}, & 1 < n \leq m, \\ \mathbf{O}, & n = 1, \end{cases} \quad (13)$$

$$V_{n-m+1}^n = W_{n-m+1}^n \cup W_{n-m+2}^n \cup \dots \cup W_{n-1}^n \cup \mathbf{O}, \quad n > m. \quad (14)$$

Here $D_{ij} = E_{ij} - E_{ji}$, \mathbf{O}^\perp is the orthogonal complement of the n -plane \mathbf{O} in \mathbb{C}^N and

$$V_l^p = \{Z \in G_n(\mathbb{C}^{n+m}) \mid \dim(Z \cap \mathbb{C}^p) \geq l\}, \quad (15)$$

$$W_l^p = V_l^p - V_{l+1}^p = \{Z \in G_n(\mathbb{C}^{n+m}) \mid \dim(Z \cap \mathbb{C}^p) = l\}. \quad (16)$$

4 Cut locus

We remember that q is in the *cut locus* \mathbf{CL}_p of $p \in \widetilde{\mathbf{M}}$ if q is the nearest point to p on the geodesic emanating from p beyond which the geodesic ceases to minimize his arc length \square . By *polar divisor* of $e_0 \in \mathbf{M}$ we mean the set $\Sigma_0 = \{\psi \in \mathbf{M} \mid (e_0, \psi) = 0\}$.

Theorem 3 *Let $\widetilde{\mathbf{M}}$ be a homogeneous manifold $\widetilde{\mathbf{M}} \approx G/K$ parametrized in the neighbourhood \mathcal{V}_0 around $Z = 0$ as in eq. (1). Then (the disjoint union)*

$$\widetilde{\mathbf{M}} = \mathcal{V}_0 \cup \Sigma_0. \quad (17)$$

Moreover, if the condition B) is true, then

$$\Sigma_0 = \mathbf{CL}_0. \quad (18)$$

A) $\text{Exp}|_o = \lambda \circ \exp|_{\mathfrak{m}}$.

B) On the Lie algebra \mathfrak{g} of G there exists an $\text{Ad}(G)$ -invariant, symmetric, non-degenerate bilinear form B such that the restriction of B to the Lie algebra \mathfrak{k} of K is likewise non-degenerate.

Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the orthogonal decomposition of \mathfrak{g} with respect to the B -form, $\text{Exp}_p : \widetilde{\mathbf{M}}_p \rightarrow \widetilde{\mathbf{M}}$ is the geodesic exponential map, $\exp : \mathfrak{g} \rightarrow G$, $o = \lambda(e)$, e is the unit element in G and λ is the projection $\lambda : G \rightarrow G/K$. Note that *the symmetric spaces have property A) and if $\widetilde{\mathbf{M}} \approx G/K$ verifies B), then it also verifies A) (cf. [1]).*

Comment 1 *The cut locus is present everywhere one speaks about coherent states.*

5 Calabi's diastasis

We remember that the *Calabi's diastasis* is expressed through the coherent states as $D(Z', Z) = -2 \log |(\underline{e}_{Z'}, \underline{e}_Z)|$ (cf. [1]). Let also $d_c([\omega'], [\omega]) = \arccos \frac{|(\omega', \omega)|}{\|\omega'\| \|\omega\|}$ denote the hermitian elliptic Cayley distance on the complex projective space.

Proposition 3 *The diastasis distance $D(Z', Z)$ between $Z', Z \in \mathcal{V}_0 \subset \widetilde{\mathbf{M}}$ is related to the geodesic distance $\theta = d_c(\iota_*(Z'), \iota_*(Z))$, where $\iota : \widetilde{\mathbf{M}} \hookrightarrow \mathbf{P}(\mathcal{L})$, by the relation*

$$D(Z', Z) = -2 \log \cos \theta.$$

If $\widetilde{\mathbf{M}}_n$ is noncompact, $\iota' : \widetilde{\mathbf{M}}_n \hookrightarrow \mathbb{CP}^{N-1,1} = SU(N, 1)/S(U(N) \times U(1))$, and $\delta_n(\theta_n)$ is the length of the geodesic joining $\iota'(Z'), \iota'(Z)$ (resp. $\iota(Z'), \iota(Z)$), then

$$\cos \theta_n = (\cosh \delta_n)^{-1} = e^{-D/2}.$$

Comment 2 *The relation (18) furnishes for manifolds of symmetric type a geometric description of the domain of definition of Calabi's diastasis: for z fixed, $z' \notin \mathbf{CL}_z$.*

6 The Euler-Poincaré characteristic, Borel-Morse cells, Kodaira embedding

Theorem 4 *For flag manifolds $\widetilde{\mathbf{M}} \approx G/K$, the following 7 numbers are equal:*

- 1) *the maximal number of orthogonal coherent vectors;*
- 2) *the number of holomorphic global section of the holomorphic line bundle \mathbf{M} ;*
- 3) *the dimension of the fundamental representation in the Borel-Weil theorem;*
- 4) *the minimal N appearing in the Kodaira embedding theorem, $\iota : \widetilde{\mathbf{M}} \hookrightarrow \mathbb{CP}^{N-1}$;*
- 5) *the number of critical points of the energy function f_H attached to a Hamiltonian H linear in the generators of the Cartan algebra of G , with unequal coefficients;*
- 6) *the Euler-Poincaré characteristic $\chi(\widetilde{\mathbf{M}}) = [W_G]/[W_H]$, $[W_G] = \text{card } W_G$, where W_G denotes the Weyl group of G ;*
- 7) *the number of Borel-Morse cells which appear in the CW-complex decomposition of $\widetilde{\mathbf{M}}$.*

Comment 3 *The Weil prequantization condition is nothing else than the condition to have a Kodaira embedding, i.e. the algebraic manifold $\tilde{\mathbf{M}}$ to be a Hodge one.*

Acknowledgments

The author expresses his thanks to Professor Heinz-Dietrich Doebner for the possibility to attend the Colloquium. Discussions during the Colloquium with Professors Martin Bordermann, Michael Forger, Joachim Hilgert, John Klauder, Askold Perelomov and Martin Schlichenmaier are acknowledged.

References

- [1] *Coherent States*, (Eds. J. R. Klauder and B. S. Skagerstam), Word Scientific, Singapore, 1985.
- [2] S. Berceanu, “The coherent states: old geometrical methods in new quantum clothes”, Bucharest preprint FT-398-1994, hep-th/9408008; S. Berceanu, “Coherent states, embeddings and transition amplitudes”, in *Quantization, Coherent States and Poisson Structures*, (Eds A. Strasburger, S. Twareque Ali, J.-P. Antoine and A. Odziejewicz), Wydawnictwo Naukowe PWN, Warsaw (in press).
- [3] S. Berceanu, “Coherent states and global Differential Geometry”, in *Quantization, coherent states, and complex structures*, (Eds. J. P. Antoine, S. Twareque Ali, W. Lisiecki, I. M. Mladenov and A. Odziejewicz), p. 131–140, Plenum, New York, 1995; S. Berceanu, “Coherent states and geodesics: cut locus and conjugate locus”, *J. Geom. Phys.* **21** 149–168 (1997); S. Berceanu, “A remark on Berezin’s quantization and cut locus”, *Rep. Math. Phys.* (in press).
- [4] S. Berceanu and A. Gheorghe, “On the construction of perfect Morse functions on compact manifolds of coherent states”, *J. Math. Phys.* **28**, 2899–2907 (1987).
- [5] S. Berceanu and L. Boutet de Monvel, “Linear dynamical systems, coherent state manifolds, flows and matrix Riccati equation”, *J. Math. Phys.* **34**, 2353–2371 (1993).
- [6] S. Berceanu, “On the geometry of complex Grassmann manifold, its noncompact dual and coherent states”, *Bull. Belg. Math. Soc.* **4** 205–243 (1997)
- [7] M. Cahen, S. Gutt and J. Rawnsley, “Quantization of Kähler manifolds” I, *J. Geom. & Phys.* **7** 45–62 (1990); II, *Trans. Math. Soc.* **337** 73–98 (1993).
- [8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. II. Interscience, New York, 1969.